Stability Analysis Using Optimization Techniques

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1. Features of Epidemiological Models

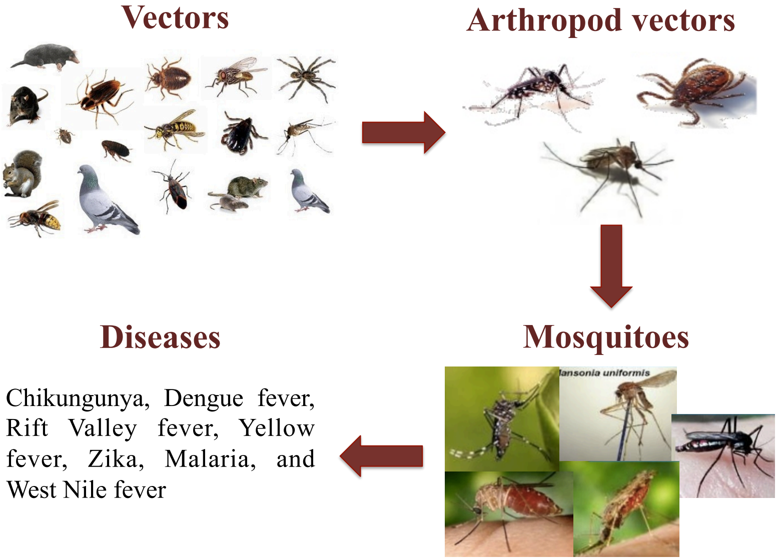
2. Stability Analysis

3. Optimization Techniques
   - Sum of Squares
   - Application of Handelman's Theorem

4. Results

Figure: Aleksandr Lyapunov (June 6, 1857 - November 3, 1918).

To build **Lyapunov functions** associated with **epidemiological models** of transmission of infectious diseases transmitted by vectors in the framework of the **optimization**.


Vectors
Arthropod vectors
Mosquitoes
uniformis
Diseases
Chikungunya, Dengue fever, Rift Valley fever, Yellow fever, Zika, Malaria, and West Nile fever

1. Nonlinear differential equations,

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$$

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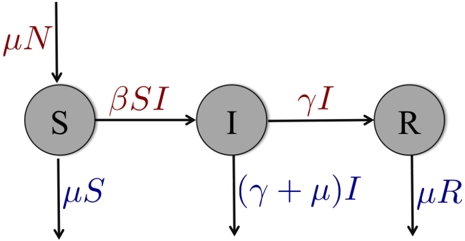
2. $\dot{\mathbf{x}}(t) = \mathit{input} - \mathit{output}$

$$\frac{dS}{dt} = \mu N - \beta SI - \mu S$$
$$\frac{dI}{dt} = \beta SI - \beta(\mu + \gamma)I$$
$$\frac{dR}{dt} = \gamma I - \mu R$$



3. Biological considerations
   - Diseases stages: susceptibles, exposed, infected and recoved
   - Interactions between populations involved in transmission process

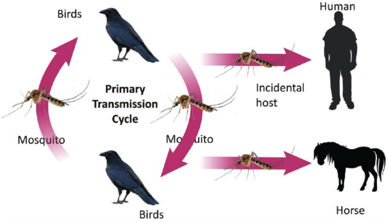


Figure: Image taken from `https://goo.gl/dWuQJp`

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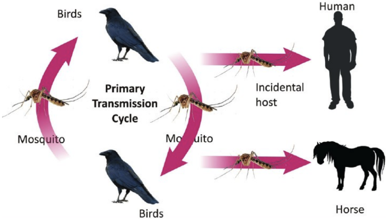


Figure: Image taken from `https://goo.gl/dWuQJp`

- Developmental stages populations

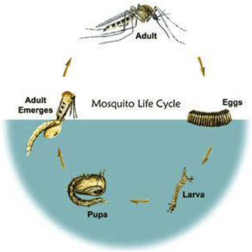
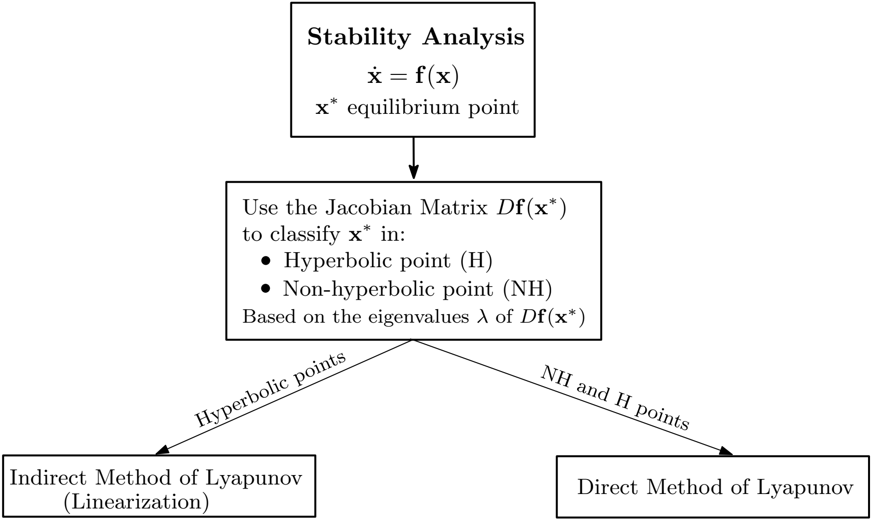


Figure: Image taken from `https://goo.gl/bVNa8l`



Stability Analysis
$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
$\mathbf{x}^*$ equilibrium point
Use the Jacobian Matrix $D\mathbf{f}(\mathbf{x}^*)$ to classify $\mathbf{x}^*$ in:
• Hyperbolic point (H)
• Non-hyperbolic point (NH)
Based on the eigenvalues λ of $D\mathbf{f}(\mathbf{x}^*)$
Hyperbolic points
NH and H points
Indirect Method of Lyapunov (Linearization)
Direct Method of Lyapunov

The following result was taken from (Khalil, 1996).

Theorem

Let $\mathbf{x}^* = \mathbf{0}$ *be an equilibrium point of* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$*. Let* $V : D \to \mathbb{R}$ *be a continuously differentiable function on a neighborhood* D *of* $\mathbf{x}^* = \mathbf{0}$, *such that*

$$V(\mathbf{0}) = 0 \text{ and } V(\mathbf{x}) > 0 \text{ in } D - \{\mathbf{0}\} \tag{1}$$

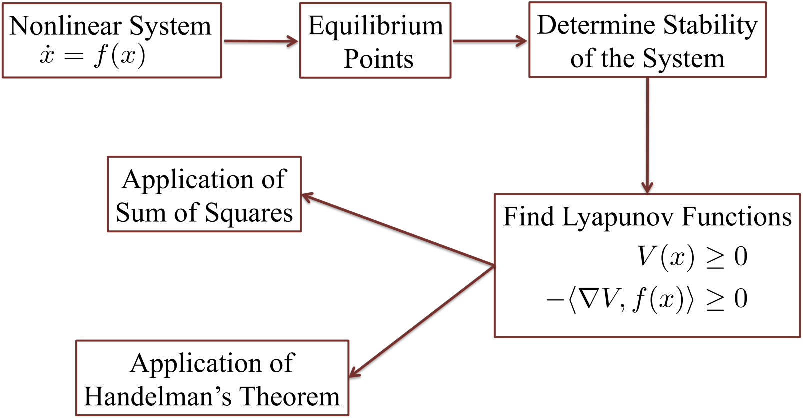
$$\dot{V}(\mathbf{x}) \leq 0 \text{ in } D \tag{2}$$

then, $\mathbf{x}^* = \mathbf{0}$ *is stable, where* $\dot{V}(\mathbf{x}) = \langle \nabla V(\mathbf{x}), \mathbf{f}(\mathbf{x}) \rangle$.

Moreover, if

$$\dot{V}(\mathbf{x}) < 0 \text{ in } D - \{\mathbf{0}\}$$

then $\mathbf{x}^* = \mathbf{0}$ *is asymptotically stable.*



Nonlinear System
$\dot{x} = f(x)$
Equilibrium Points
Determine Stability of the System
Find Lyapunov Functions
$V(x) \geq 0$
$-\langle \nabla V, f(x) \rangle \geq 0$
Application of Sum of Squares
Application of Handelman's Theorem

Exponentially Stable Nonlinear Systems Have Polynomial Lyapunov Functions on Bounded Regions

Matthew M. Peet, *Member, IEEE*

Matthew M. Peet (S'02–M'06) received the B.S. degrees in physics and in aerospace engineering from the University of Texas at Austin in 1999 and the M.S. and the Ph.D. degree in aeronautics and astronautics from Stanford University, Stanford, CA, in 2001 and 2006, respectively.

Theorem (Peet, 2009)

Consider the system $\dot{x}(t) = f(x(t))$ where $D^\alpha f \in C_1^2(\mathbb{R}^n)$ for all $\alpha \in Z^n$. Suppose there exist constants $\mu, \delta, r > 0$ such that

$$||Ax_0(t)||_2 \leq \mu ||x_0||_2 e^{-\delta t}$$

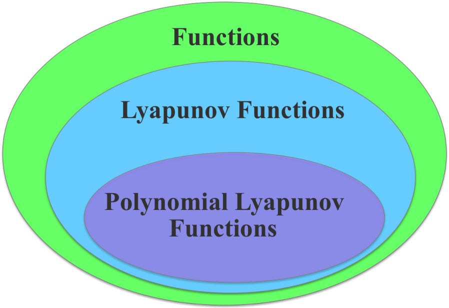
for all $t \geq 0$ and $||x_0||_2 \leq r$.

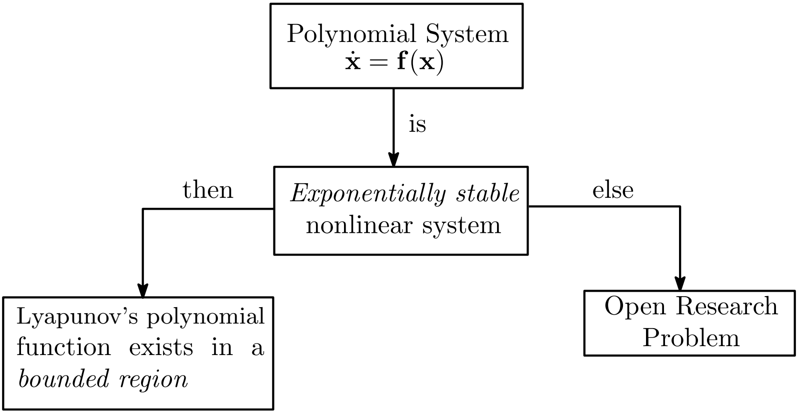
Then, there exists a **polynomial** *$v : \mathbb{R}^n \to \mathbb{R}$ and constants $\alpha, \beta, \gamma, \mu > 0$ such that*

$$\alpha ||x||_2^2 \leq v(x) \leq \beta ||x||_2^2$$

$$\nabla v(x)^T f(x) \leq -\gamma ||x||_2^2$$

- As a sufficient condition, we demand from function f to be $n+2$-times continuously differentiable in order to satisfy the conditions of the theorem.
- As a consequence of this theorem, we have a corollary that tells us that **ordinary differential equations defined by polynomials have Lyapunov polynomial functions**.





Polynomial System
$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
is
Exponentially stable nonlinear system
then
else
Lyapunov's polynomial function exists in a *bounded region*
Open Research Problem

Result: SIR model is exponentially stable

$$\dot{s} = \mu - \beta s i - \mu s$$
$$\dot{i} = \beta s i - (\mu+\gamma) i$$
$$\dot{r} = \gamma i - \mu r$$

$\longrightarrow$

$$\dot{s} = \mu - \beta s i - \mu s$$
$$\dot{i} = \beta s i - (\mu+\gamma) i$$

$\longrightarrow$

- $R_0 = \frac{\beta}{\mu+\gamma}$
- $E_0 = (1,0)$
- $E_1 = (s^*, i^*)$

where $s^* = \frac{1}{R_0}$, and $i^* = \frac{\mu}{\beta}(R_0 - 1)$

$\downarrow$

$$J(s,i) = \begin{bmatrix} -\beta i - \mu & \beta s \\ \beta i & \beta s - (\mu+\gamma) \end{bmatrix}$$

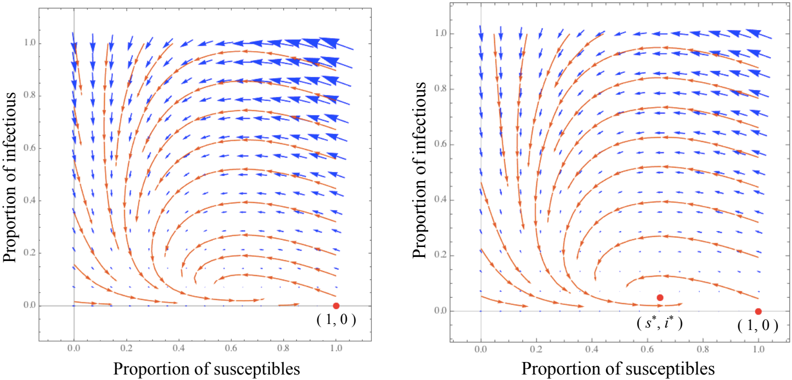
$$p(\lambda) = (-\beta i - \mu - \lambda)(\beta s - (\mu+\gamma) - \lambda) + \beta^2 s i$$

$\xrightarrow{E_0}$

- $p(\lambda) = (-\mu - \lambda)(\beta - (\mu+\gamma) - \lambda) = 0.$
- $\lambda_1 = -\mu < 0$
- $\lambda_2 = \beta - (\mu+\gamma) < 0$ iff $R_0 < 1.$

$\xrightarrow{E_1}$

- $p(\lambda) = \lambda^2 + \mu R_0 \lambda + \mu(\mu+\gamma)(R_0 - 1)$
- $\lambda_i < 0$, for $i = 1,2$ iff $R_0 > 1.$



(a) $R_0 \leq 1$ (b) $R_0 > 1$

Figure: In (a) $\mu = 0.2$, $\beta = 0.5$, $\gamma = 0.8$, $R_0 = 0.5$, in (b) $\mu = 0.08$, $\beta = 0.9$, $\gamma = 0.5$, $R_0 = 1.55$

Questions

1. How to construct these polynomials?
2. What should be the degree of these polynomials?
3. How can we verify that a polynomial is positive?

Figure: David Hilbert (23 January 1862 - 14 February 1943)

Mathematical Problems

1. Cantor's problem of the cardinal number of the continuum
2. The compatibility of the arithmetical axioms
3. The equality of two volumes of two tetrahedra of equal bases and equal altitudes*
4. Problem of the straight line as the shortest distance between two points
5. Lie's concept of a continuous group of transformations without the assumption of the differentiability of the functions defining the group
6. Mathematical treatment of the axioms of physics
7. Irrationality and transcendence of certain numbers*

Mathematical Problems

8. Problems of prime numbers
9. Proof of the most general law of reciprocity in any number field
10. Determination of the solvability of a diophantine equation*
11. Quadratic forms with any algebraic numerical coefficients
12. Extension of Kroneker's theorem on abelian fields to any algebraic realm of rationality
13. Impossibility of the solution of the general equation of the 7-th degree by means of functions of only two arguments
14. Proof of the finiteness of certain complete systems of functions*
15. Rigorous foundation of Schubert's enumerative calculus
16. Problem of the topology of algebraic curves and surfaces

Mathematical Problems [1]

17. Express a nonnegative rational function as quotient of sums of squares.*
18. Building up of space from congruent polyhedra*
19. Are the solutions of regular problems in the calculus of variations always necessarily analytic?*
20. The general problem of boundary values*
21. Proof of the existence of linear differential equations having a prescribed monodromic group*
22. Uniformization of analytic relations by means of automorphic functions*
23. Further development of the methods of the calculus of variations

[1] The asterisk means that the problem is solved. Furthermore, the problem that we want to solve is related to the problem number 17.

The following result was taken from (Kamyar, 2015).

Theorem (Artin's theorem)

A polynomial $f \in \mathbb{R}[x]$ satisfies $f(x) \geq 0$ on $\mathbb{R}^n$ if and only if there exist Sum of Squares (SOS) polynomials N and $D \neq 0$ such that $f(x) = \frac{N(x)}{D(x)}$.

SOME CONCRETE ASPECTS OF HILBERT'S 17TH PROBLEM

Bruce Reznick

University of Illinois

This paper is dedicated to the memory of Raphael M. Robinson and Olga Taussky Todd.

Definition

A convex program problem is an optimization problem of the type

$$\min\ g(x)$$
$$\text{subject to } x \in X$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a convex function, and the feasible set $X \subseteq \mathbb{R}^n$ is a convex set.

Definition

A semidefinite program (SDP) problem is a convex program problem of the form

$$\min\ c^T x$$
$$\text{s.t. } A_0 + \sum_{i=1}^{m} x_i A_i \succeq 0$$

where $x \in \mathbb{R}^m$ is the decision variable, and $c \in \mathbb{R}^m$ and the $m+1$ symmetric $n \times n$ matrices A_i are given data of the problem.

Consider the linear system

$$\dot{x}(t) = Ax(t)$$

Let a quadratic Lyapunov function

$$V(x) = x^T P x \tag{3}$$

where,

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T (A^T P + PA) x \end{aligned} \tag{4}$$

From (3) and (4) we formulate the semidefinite programming problem

$$\begin{aligned} P &\succ 0 \\ A^T P + PA &\prec 0 \end{aligned} \tag{5}$$

We solve (5) for the linear system

$$\dot{x}(t) = \begin{bmatrix} -1 & 4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- $P \succ 0$ iff $x^T P x > 0$. In fact,

$$[x \;\; y] \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = p_{11}x^2 + 2p_{12}xy + p_{22}y^2 > 0$$

  We assume $p_{12} = 0$, thus $x^T P x = p_{11}x^2 + p_{22}y^2 > 0$ iff $p_{11},\, p_{22} > 0$.

- $A^T P + PA \prec 0$ iff

$$
\begin{aligned}
&[x \;\; y] \begin{bmatrix} -2p_{11} & 4p_{11} - p_{22} \\ 4p_{11} - p_{22} & -2p_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= -2p_{11}x^2 + (8p_{11} - 2p_{22})xy - 2p_{22}y^2 < 0
\end{aligned}
$$

If we consider $(8p_{11} - 2p_{22}) = 0$, then $p_{22} = 4p_{11}$.

Thus $P = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ is a solution of (5) and $V(x) = x^2 + 4y^2$.

Definition (Parrilo, 2000)

A multivariate polynomial $p(x_1, \cdots, x_n) := p(x)$ is a sum of squares, if there exist polynomials $q_1(x), \cdots, q_m(x)$ such that

$$p(x) = q_1^2(x) + q_2^2(x) + \cdots + q_m^2(x)$$

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Theorem (Parrilo, 2000, 2003)

A multivariate polynomial $p(x)$ in n variables and of degree $2d$ is a sum of squares if and only if there exists a positive semidefinite matrix Q such that

$$p(x) = z^T Qz, \tag{6}$$

where z is the vector of monomials of degree up to d

$$z^T = [1, x_1, x_2, \cdots, x_n, x_1x_2, \cdots, x_n^d] \tag{7}$$

Definition of Lyapunov function

$$V(x) > 0$$
$$-\langle \nabla V(\mathbf{x}), \mathbf{f}(\mathbf{x}) \rangle > 0$$

Relaxation of constraints

$$V(x) \text{ is a SOS}$$
$$-\langle \nabla V(x), \mathbf{f}(\mathbf{x}) \rangle \text{ is a SOS}$$

$$V(x) - \epsilon \sum_{i=1}^{n} x_i^q \text{ is a SOS}$$
$$-\langle \nabla V(x), \mathbf{f}(\mathbf{x}) \rangle - \epsilon \sum_{i=1}^{n} x_i^q \text{ is a SOS}$$

where ϵ is a fixed small positive number, and q is the degree of Lyapunov function, V.

1. Define the degree of Lyapunov function, $2d$.
2. Define the vector of monomials z of degree up to d.
3. Express the Lyapunov function as a quadratic form, i.e. $V(x) = z^T Qz$.
4. If in the representation above Q is positive semidefinite, then $V(x)$ is also positive semidefinite.

Theorem

The following statements are equivalent:

1. *The symmetric matrix A is positive semidefinite.*
2. *All eigenvalues of A are nonnegative.*
3. *All the principal minors of A are nonnegative.*
4. *There exists B such that $A = B^T B$.*

Theorem

Let $A \in \mathbb{R}^{n \times n}$ a symmetric matrix. Then A is positive semidefinite if and only if all the coefficients of its characteristic polynomial

$$p(\lambda) = \det(\lambda I_n - A) = \lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0 \tag{8}$$

have alternating signs, i.e., $(-1)^{n-i} p_i \geq 0$ for all $i = 1, \cdots, n$.

Result: Lyapunov function for *sir* model

Consider the system *sir* normalized, i.e, $s + i + r = 1$.

$$\frac{ds}{dt} = \mu - \beta si - \mu s$$
$$\frac{di}{dt} = \beta si - (\mu + \gamma)i$$
$$\frac{dr}{dt} = \gamma i - \mu r$$

$$\begin{aligned} \frac{ds}{dt} &= \mu - \beta si - \mu s \\ \frac{di}{dt} &= \beta si - (\mu + \gamma)i \end{aligned} \tag{9}$$

The system (9) has two equilibrium points:

- The disease-free point, $E_0 = (1, 0)$, and
- The endemic equilibrium point, $E_1 = (s^*, i^*)$, where $s^* = \frac{1}{R_0}$, and $i^* = \frac{\mu}{\beta}(R_0 - 1)$, with $R_0 = \frac{\beta}{\mu+\gamma}$.

Moving the disease-free point $E_0 = (1, 0)$ to the origin, the system (9) becomes:

$$\begin{aligned} \dot{x_1} &= \mu - \beta(1 + x_1)x_2 - \mu(1 + x_1) \\ \dot{x_2} &= \beta(1 + x_1)x_2 - (\mu + \gamma)x_2 \end{aligned} \tag{10}$$

where $x_1 = s - 1$, and $x_2 = i$.

We solve the optimization problem without objective function

$$\begin{aligned} V(x_1, x_2) - \epsilon(x_1^2 + x_2^2) &\text{ is a SOS} \\ -\langle \nabla V(x_1, x_2), (f(x_1, x_2)) \rangle - \epsilon(x_1^2 + x_2^2) &\text{ is a SOS} \end{aligned}$$

To look for a Lyapunov function, we will use the general expression of a polynomial in x_1 and x_2 of degree two with neither constant nor linear terms.

$$
\begin{aligned}
V(x_1, x_2) &= [x_1 \;\; x_2] \begin{bmatrix} q_{11} - \epsilon & q_{12} \\ q_{12} & q_{22} - \epsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= (q_{11} - \epsilon)x_1^2 + 2q_{12}x_1x_2 + (q_{22} - \epsilon)x_2^2
\end{aligned}
$$

Assume $q_{12} = 0$. The matrix $Q = \begin{bmatrix} q_{11} - \epsilon & 0 \\ 0 & q_{22} - \epsilon \end{bmatrix}$

is semidefinite positive iff $q_{11} - \epsilon \geq 0$, and $q_{22} - \epsilon \geq 0$.

For the derivative, we obtain after some algebra that we have $\dot{V}(x_1, x_2) = -w^T R w$, with the vector $w = [x_1 \;\; x_2 \;\; x_1 x_2]$.

The expression for the matrix R is

$$\begin{bmatrix} 2(q_{11}-\epsilon)\mu & (q_{11}-\epsilon)\beta & (q_{11}-\epsilon)\beta \\ (q_{11}-\epsilon)\beta & 2(q_{22}-\epsilon)(\mu+\gamma)(1-R_0) & -(q_{22}-\epsilon)\beta \\ (q_{11}-\epsilon)\beta & -(q_{22}-\epsilon)\beta & 0 \end{bmatrix}$$

here ϵ is a fixed small positive number

In general, we found

$$V(s,i) = q_{11}(s-1)^2 + q_{22}i^2$$

where

$$q_{11} = \epsilon \text{ and } q_{22} = \frac{\epsilon(\mu+\gamma)}{(\gamma+1)}$$

with $0.06 \leq \mu \leq 0.3$, $0 \leq \beta \leq 1$, $0.5 \leq \gamma \leq 1.75$ and $\epsilon > 0$

SIMULATIONS

$$V(s,i) = q_{11}(s-1)^2 + q_{22}i^2$$

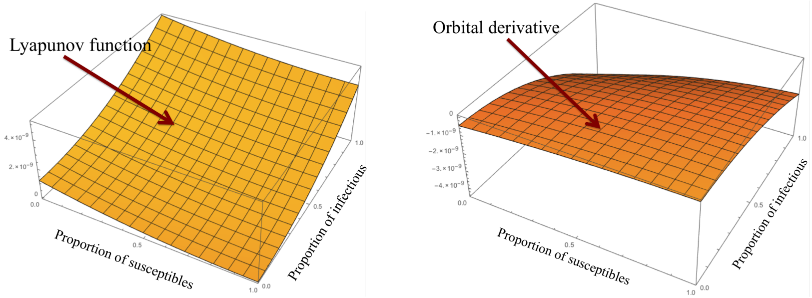


Figure: $\mu = 0.2$, $\beta = 0.5$, $\gamma = 0.8$, $R_0 = 0.5$, $q_{11} = 1.201 \times 10^{-4}$, and $q_{22} = 5.666 \times 10^{-5}$

Dengue transmission model

$$
\begin{aligned}
\frac{dm_e}{dt} &= b\beta_m h_i(1 - m_e - m_i) - (\theta_m + \mu_m)m_e \\
\frac{dm_i}{dt} &= \theta_m m_e - \mu_m m_i \\
\frac{dh_s}{dt} &= \mu_h - b\beta_h m_i h_s - \mu_h h_s \\
\frac{dh_e}{dt} &= b\beta_h m_i h_s - (\theta_h + \mu_h)h_e \\
\frac{dh_i}{dt} &= \theta_h h_e - (\gamma_h + \mu_h)h_i
\end{aligned}
$$

The disease-free point, $P_0 = (0, 0, 1, 0, 0)$.

Moving the disease-free point P_0 to the origin:

$$\frac{dx_1}{dt} = b\beta_m x_5(1 - x_1 - x_2) - (\theta_m + \mu_m)x_1$$
$$\frac{dx_2}{dt} = \theta_m x_1 - \mu_m x_2$$
$$\frac{dx_3}{dt} = \mu_h - b\beta_h x_2(x_3 + 1) - \mu_h(x_3 + 1)$$
$$\frac{dx_4}{dt} = b\beta_h x_2(x_3 + 1) - (\theta_h + \mu_h)x_4$$
$$\frac{dx_5}{dt} = \theta_h x_4 - (\gamma_h + \mu_h)x_5$$

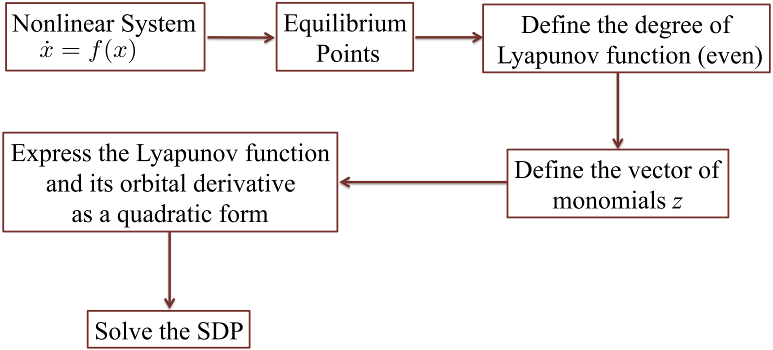
In general, we found

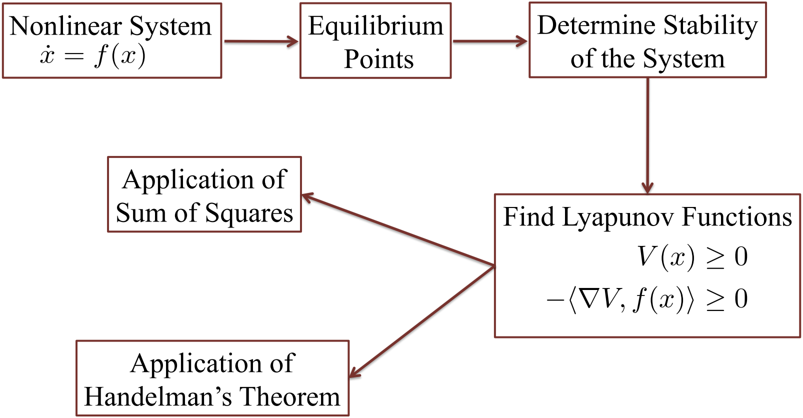
$$V(m_e, m_i, h_s, h_e, h_i) = q_{11} m_e^2 + q_{22} m_i^2 + q_{33}(h_s - 1)^2 + q_{44} h_e^2 + q_{55} h_i^2$$

where

$$\begin{aligned}
q_{11} &= \epsilon \\
q_{22} &= \frac{\lambda}{\sqrt{(\theta_m + \mu_m)}} + \epsilon \\
q_{33} &\leq \frac{4\mu_h \mu_m}{b^2 \beta_h^2}(q_{22} - \epsilon) + \epsilon \\
q_{44} &\leq \frac{4\mu_m(\theta_h + \mu_h)}{b^2 \beta_h^2}(q_{22} - \epsilon) + \epsilon \\
q_{55} &\leq \frac{4(\theta_h + \mu_h)(\gamma_h + \mu_h)}{\theta_h^2}(q_{44} - \epsilon) + \epsilon
\end{aligned}$$

with $\epsilon > 0$


Nonlinear System
$\dot{x} = f(x)$
Equilibrium Points
Define the degree of Lyapunov function (even)
Define the vector of monomials z
Express the Lyapunov function and its orbital derivative as a quadratic form
Solve the SDP



Nonlinear System
$\dot{x} = f(x)$
Equilibrium Points
Determine Stability of the System
Find Lyapunov Functions
$V(x) \geq 0$
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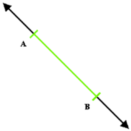
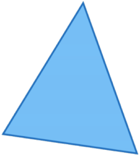
Application of Sum of Squares
Application of Handelman's Theorem

Polytopes

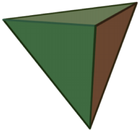
1-símplex

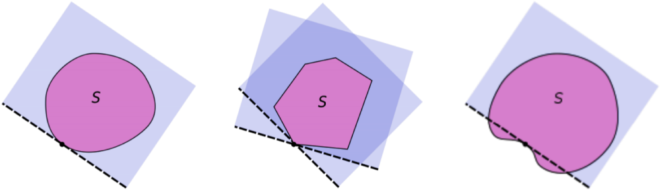


2-símplex



3-símplex





Theorem (Handelman's theorem)

Given $w_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}$, define the polytope

$$\Gamma^K := \{x \in \mathbb{R}^n \,.\, w_i^T x + u_i \geq 0, \;\; i = 1, ..., K\}. \tag{11}$$

If a polynomial $f(x) > 0$ on Γ^K, then there exist $b_\alpha \geq 0$, $\alpha \in \mathbb{N}^K$ such that for some $d \in \mathbb{N}$,

$$f(x) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ \alpha_1 + \cdots + \alpha_K \leq d}} b_\alpha (w_1^T x + u_1)^{\alpha_1} \cdots (w_K^T x + u_K)^{\alpha_K}. \tag{12}$$

This theorem was taken from (Kamyar, 2014).

$$\gamma^* = \max_{\gamma, c_\beta \in \mathbb{R}} \gamma$$

subject to

$$\begin{bmatrix} \sum_{\alpha \in E_d} c_\beta x^\beta - \gamma x^T x & 0 \\ 0 & -\langle \nabla \sum_{\alpha \in E_d} c_\beta x^\beta, f(x) \rangle - \gamma x^T x \quad 0 \end{bmatrix} \geq 0$$

for all $x \in D$.

$$(13)$$

Conditions (1) and (2) of Direct method of Lyapunov hold if and only if there exist $d \in \mathbb{N}$ such that $\gamma^* > 0$.

How can we apply Handelman's theorem to solve (13)?

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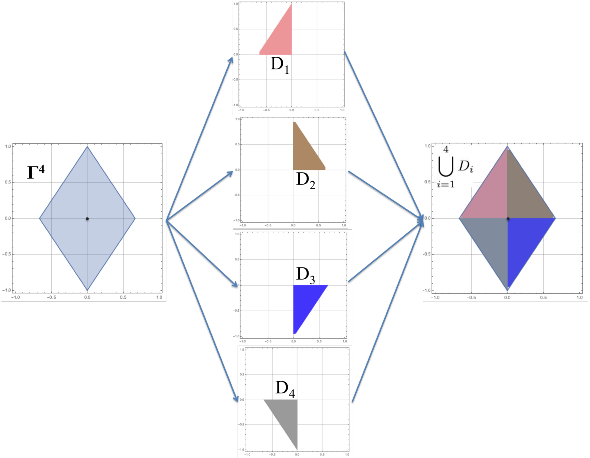


The equation (13) becomes a new linear program problem:

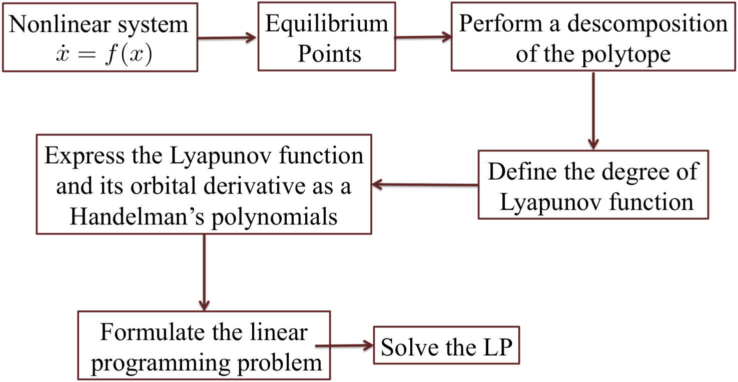
$$
\begin{aligned}
\max_{\gamma\in\mathbb{R}, b_i\in\mathbb{R}^{N_i}, c_i\in\mathbb{R}^{M_i}} \gamma& \\
\text{subject to}& \\
b_1 \geq \mathbf{0} &\quad \text{for} \quad i = 1, \cdots, L \\
c_1 \leq \mathbf{0} &\quad \text{for} \quad i = 1, \cdots, L \\
R_i(b_i, d) = \mathbf{0} &\quad \text{for} \quad i = 1, \cdots, L \\
H_i(b_i, d) \geq \gamma\mathbf{1} &\quad \text{for} \quad i = 1, \cdots, L \\
H_i(c_i, d + d_f - 1) \leq -\gamma\mathbf{1} &\quad \text{for} \quad i = 1, \cdots, L \\
G_i(b_i, d) = F_i(c_i, d + d_f - 1) &\quad \text{for} \quad i = 1, \cdots, L \\
J_{i,k}(b_i, d) = J_{j,l}(b_i, d) &\quad \text{for} \quad i, j = 1, \cdots, L, \ i \neq j, \\
&\quad k, l \in \{1, \cdots, m_i\}
\end{aligned}
\tag{14}
$$

where

- $R_i(b_i, d)$ is the vector of coefficients of monomials of $V_i(x)$ which are nonzero at the origin.
- $H_i(b_i, d)$ is the vector of coefficients of square terms of $V_i(x)$.
- $G_i(b_i, d)$ is the vector of coefficients of $\langle \nabla V_i(x), f(x) \rangle$.
- $F_i(b_i, d)$ is the vector of coefficients of $V_i(x)$.
- $J_{i,k}(b_i, d)$ is the vector of coefficients of (12), such that $\alpha_K = 0$.

This result was taken from (Kamyar, 2014).

Application of Handelman's theorem



We found robust Lyapunov functions to test the asymptotic stability of disease-free equilibrium points in some models simulating the transmission of mosquito-borne infectious diseases.

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