Lagrangian Formulation of Elastic Wave Equation on Riemannian Manifolds

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- *Based on least action principles (Calculus of variations)*
- *Allows to work with different fields, such as the electromagnetic field in one simple formulation*
- *It only considers the forces that give rise to motions*
- *Give rise to nondynamical symmetries because of the way in which we formulate the action*

Lagrangian formulation of mechanics

- The action
- Examples

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Lagrangian formulation of mechanics

- The action

$$\mathcal{L}(x, \dot{x}, t) : \mathcal{C} \to \mathcal{R}$$

and the action is

$$S = \int_{t_1}^{t_2} \mathcal{L} dt.$$

We need to solve the problem

$$\delta S = 0.$$

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which gives the equations

$$\begin{aligned} \ddot{x} &= 0 \\ \ddot{y} &= -g. \end{aligned}$$

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Consider

$$\mathcal{L}(r, \dot{r}) = \frac{1}{2} m \left[\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{\phi}^2 sin^2(\theta) \right] - V(r) \quad (1)$$

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we get the system of equations

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2 - \frac{1}{m}\frac{dV}{dr} \\ \ddot{\theta} &= -\frac{2}{r}\dot{r}\dot{\theta} + \sin\theta\cos\theta\dot{\phi}^2 \\ \ddot{\phi} &= -\frac{2}{r}\dot{r}\dot{\phi} - 2\cot\theta\dot{\theta}\dot{\phi} \end{aligned}$$

in particular, for $\theta = \frac{\pi}{2}$

$$\begin{aligned} \ddot{r} &= r\dot{\phi}^2 - \frac{1}{m}\frac{dV}{dr} \\ \ddot{\theta} &= 0 \\ \ddot{\phi} &= -\frac{2}{r}\dot{r}\dot{\phi} \end{aligned}$$

Lagrangian formulation of mechanics

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Consider a scalar field ϕ, and the Lagrangian

$$\mathcal{L}(\phi, \dot{\phi}) = \frac{1}{2} g^{\mu\nu} \left(\partial_\mu \phi\right) \left(\partial_\nu \phi\right) - \frac{1}{2} m^2 \phi^2 - V\left(\phi\right)$$

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The Euler-Lagrange equations for this field are

$$\frac{\partial \mathcal{L}}{\partial \phi^r} - \frac{\partial}{\partial q^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi^r_{,\mu}} \right) = 0$$

we get

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$$\left(\Box + m^2\right) \phi = -\frac{\partial V}{\partial \phi},$$

where

$$\Box \equiv \frac{\partial^2}{\partial t^2} - \nabla^2$$

Lagrangian Formulation on Riemannian Manifolds

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Lagrangian Formulation on Riemannian Manifolds

- Let (M, g), (N, G) be Riemannian manifolds, $\phi \in \mathcal{F}(M)$ and $\psi : M \to N$. The Lagrangian is a function

$$\mathcal{L} : TN \to \mathcal{R},$$

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- If the integral is stationary, then it satisfies the Euler-Lagrange equations on the manifold

$$\sum_{k=1}^{m} \left(\frac{\partial \mathcal{L}}{\partial(\psi^i_{;k})} \right)_{;k} = \frac{\partial \mathcal{L}}{\partial \psi^i}$$

Lagrangian Formulation for an Elastic Space of Configurations

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Lagrangian Formulation for an Elastic Space of Configurations

- Let us consider a body manifold $\mathcal{B}$ with an atlas (ψ_i, U_i) where $\mathcal{B} \subset \mathcal{R}^n$ and $\psi_i(U_i) \subset \mathcal{R}^n$; regard this manifold as the undeformed state of any elastic medium. Let $\mathcal{S}$ be ambient manifold with an atlas $(\phi(U_i), \theta_i)$ and $\phi : \mathcal{B} \to \mathcal{S}$ be a configuration of $\mathcal{B}$ into $\mathcal{S}$.
  We can consider the phase space $(\phi_t(X), T_x\phi_t(U_i))$ for $X \in U_i$ and $x = \phi_t(X)$
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- After a motion, say $\phi(X, t) = x + \mathbf{u}(\mathbf{x}, \mathbf{t})$, where $\mathbf{u}$ is the small displacement vector field on $\mathcal{S}$, we can see that the strain tensor is given by

$$\varepsilon_{ij}(x)\ dx^i \otimes dx^j = \frac{1}{2}\left\{\phi^* ds(x)^2 - ds(x)^2\right\},$$

  and after calculations on this expression we get

$$\varepsilon_{ij} = \frac{1}{2}(\mathcal{L}_u g)_{ij},$$

  where $(\mathcal{L}_u g)_{ij} = \frac{1}{2}\left(g_{il} \bigtriangledown_j u^l + g_{lj} \bigtriangledown_i u^l\right)$, are the components of the Lie derivative of the metric with respect to the displacement vector field.

Lagrangian Formulation for an Elastic Space of Configurations

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Consider the existence of the following functions and vector fields

- $e(x,t)$, internal energy functional
- $\vec{b}(x,t)$, external force vector field,
- $\mathbf{t}(\mathbf{x},\mathbf{t},\tilde{\mathbf{n}})$, traction for which exists a two-tensor σ such that $\mathbf{t}(\mathbf{x},\mathbf{y},\tilde{\mathbf{n}}) = \sigma(\mathbf{x},\mathbf{t})\cdot\tilde{\mathbf{n}}$, where $\vec{n}$ is the normal outward to the manifold at every point.

Since changes of the metric on the manifold $\mathcal{S}$ affect the accelerations of the particles, the internal energy must depend parametrically on the metric g, and if we have balance of energy

$$\frac{d}{dt}\int_{\phi(U)}\rho\left(e+\frac{1}{2}<\vec{v},\vec{v}>\right)dv = \int_{\phi(U)}\rho<\vec{b},\vec{v}>dv + \int_{\partial\phi(U)}<\mathbf{t},\tilde{\mathbf{v}}>\mathbf{da},$$

it can be proved that

$$\sigma_{ij} = 2\rho\frac{\partial e}{\partial g.}$$

Lagrangian Formulation for an Elastic Space of Configurations

Consider the Lagrangian $\mathbb{L} : T\mathcal{S} \to \mathcal{R}$ given by

$$\mathbb{L}(x, \vec{v}) = \frac{1}{2} < \vec{v}, \vec{v} > -e(x, t, g),$$

and the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathbb{L}}{\partial \vec{v}^{\mu}}\right) = \frac{\partial \mathbb{L}}{\partial x^{\mu}}.$$

Lagrangian Formulation for an Elastic Space of Configurations

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then we have the system of equations

$$g_{\mu i}a^i = -\left(\frac{\partial e}{\partial x^\mu} + \frac{\sigma_{ij}}{2\rho}\cdot\frac{\partial g_{ij}}{\partial x^\mu}\right). \qquad (2)$$

If we consider small perturbations, on which Hook's law is valid, we can take the energy functional to be

$$e(x,t,g) = < C\varepsilon, \varepsilon >$$

which clearly depends on the metric; and $\sigma = C\varepsilon$ then we rewrite equation the above equation as

$$\rho g_{\mu i}a^i = -\left(\frac{\partial < C\varepsilon, \varepsilon >}{\partial x^\mu} + \frac{\sigma}{2\rho}\cdot\frac{\partial g_{ij}}{\partial x^\mu}\right), \qquad (3)$$

after some manipulations and the use of Leibnitz's rule we have

$$\rho g_{\mu i}a^i = -C_{ijkl}\left[\frac{\partial}{\partial x^\mu}\left\langle \varepsilon_{kl}, \varepsilon_{ij} + \frac{g_{ij}}{2}\right\rangle - \frac{1}{2}\left\langle \frac{\partial \varepsilon_{kl}}{\partial x^\mu}, g_{ij}\right\rangle\right]. \qquad (4)$$

Since we are assuming the time invariance of the medium, the causality of the wave motion is going to be taken into account. Let ω be the time-Fourier parameter for the field $\mathbf{u}$ and denote $\hat{\mathbf{u}}(x,\omega)$, $\hat{\varepsilon_{ij}}$ the associated fields after Fourier transform on the time variable, then we have the equation

$$-\omega^2\rho g_{\mu i}(x)\hat{\mathbf{u}} + C_{ijkl}\frac{\partial}{\partial x^\mu}\left\langle \hat{\varepsilon_{kl}}, \hat{\varepsilon_{ij}} + \frac{g_{ij}}{2}\right\rangle - \frac{1}{2}C_{ijkl}\left\langle \frac{\partial \hat{\varepsilon_{kl}}}{\partial x^\mu}, g_{ij}\right\rangle = 0 \qquad (5)$$

Lagrangian Formulation for an Elastic Space of Configurations

Since we are interested in a particular direction of propagation, say a geodesic one, in geodesic coordinates suppose $j = 3$; we get the following system of equations

$$
\begin{aligned}
-\omega^2 \rho g_{\mu 3} \hat{\mathbf{u}}^3 + \frac{\partial}{\partial x^\mu} \left\langle \hat{\sigma}_{33}, \hat{\varepsilon}_{33} + \frac{g_{33}}{2} \right\rangle - \frac{1}{2} \left\langle \frac{\partial}{\partial x^\mu} \hat{\sigma}_{33}, g_{33} \right\rangle &= 0 \\
-\omega^2 \rho g_{\mu\nu} \hat{\mathbf{u}}^\nu + \frac{\partial}{\partial x^\mu} \left\langle \hat{\sigma}_{\nu 3}, \hat{\varepsilon}_{\nu 3} \right\rangle &= 0. \quad v = 1, 2
\end{aligned}
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- To perform elastic one-way wave equations on a Riemannian manifold in local coordinates: Flux “normalization” and subprincipal symbol, self-adjoint form.
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- To perform elastic one-way wave equations on a Riemannian manifold in local coordinates: Flux "normalization" and subprincipal symbol, self-adjoint form.
- Tensor upward/downward continuation with a Riemannian metric.
- To consider the elastic wave equation for the metric, resulting from the Einstein-Hilbert action on the manifold $\mathcal{S}$.

Some references

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